THE CALGARY MATHEMATICAL ASSOCIATION

39th JUNIOR HIGH SCHOOL MATHEMATICS CONTEST

APRIL 29, 2015

NAME: ______________________________ GENDER: ________
PLEASE PRINT (First name Last name) (optional)

SCHOOL: ______________________________ GRADE: ________
(9,8,7,...)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given. PART A has a total possible score of 45 points.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct. PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale. They are intended as visual hints only.
- Hint: Read all the problems and select those you have the best chance to solve first. You may not have time to solve all the problems.

MARKERS' USE ONLY	
PART A ________×5	
B1	
B2	
B3	
B4	
B5	
B6	
TOTAL (max: 99)	

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.

THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

PART A: SHORT ANSWER QUESTIONS (Place answers in the boxes provided)

A1 At a bus station, a bus leaves at 8:00 am and a new bus leaves every 7 minutes after that. At what time does the first bus after 9:00 am leave?

A1: 9:03 AM (or 9:03)

Solution: The following buses leave at 8:07, 8:14, 8:21, 8:28, 8:35, 8:42, 8:49, 8:56, 9:03.

A2 If we mix one litre of lemonade that contains 4% lemon with two litres of lemonade that contains 10% lemon, what is the percentage of lemon in the resulting three litre mixture?

A2: 8%

Solution: There is 0.24 litres of lemon in the three litre mixture. Thus, the mixture contains $0.24/3 = 0.08$ or 8% lemon.

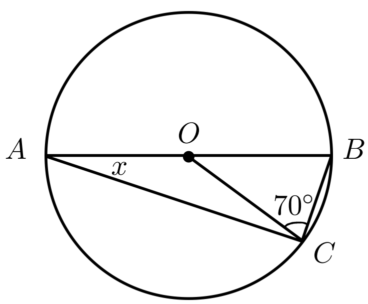
A3 At the swimming pool last week, on each day there were ten fewer people than twice the number of people on the previous day. There were 130 people at the pool on Friday. How many people were at the pool on the previous Tuesday?

A3: 25

Solution: On Thursday there were $(130 + 10)/2 = 70$ people. On Wednesday there were $(70 + 10)/2 = 40$ people. On Tuesday there were $(40 + 10)/2 = 25$ people.

A4 Given the circle below with centre O, find the angle x in degrees.

A4: 20°



Since $OA = OB = OC$, triangles OAC and OBC are isosceles. Hence, $\angle OCA = x$ and $\angle OBC = 70°$. As $\angle COB = 40°$, we have $\angle AOC = 140°$. Thus, $x + x = 40°$ implying $x = 20°$.

A5 Firmamint boxes of chocolates contain 17 with hard centres and 5 with soft centres. Sweetart boxes contain 7 with hard centres and 11 with soft centres. If I buy one Firmamint box, how many Sweetart boxes must I buy so that the total number of hard centres is equal to the total number of soft centres?

A5: 3

Solution 1: The difference of 12 between the numbers of hard centres and soft centres in the Firmamint box must be made up by the Sweetart boxes, which can reduce the difference by 4 per box, so 3 boxes of Sweetarts are required.

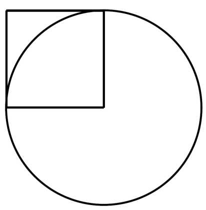
Solution 2: f Firmamint boxes contain $17f$ hard and $5f$ soft, while s Sweetart contain $7s$ hard and $11s$ soft. If $17f + 7s = 5f + 11s$, then $12f = 4s$ and you need exactly 3 times as many Sweetart as Firmamint.

A6 Sagal and Xi leave home at the same time to walk to the park which is 6 km away. Sagal walks at 1 km/hr for 2 km, then at 2 km/hr for 4 km. Xi walks at 1 km/hr for 4 km, then at 2 km/hr for 2 km. Sagal arrives at the park at noon. At what time does Xi arrive?

A6
1:00 PM (or 1:00) (or 13:00)

A7 In the following figure the square has one corner in the centre of the circle and two sides are tangent to the circle. How many times larger is the area of the circle than the area of the square?

A7
π (or 3.14)



Solution: The answer is π and can be calculated assuming that the radius is r and then dividing the area of the circle by the area of the square, $\frac{\pi r^2}{r^2}$.

A8 Below, the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ are to be filled into the nine smaller squares so that every number is used exactly once. If the sum of each row and the sum of each column is at most 15, what must the value of x be?

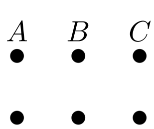
A8
9

	x	
		7
8		

Solution: Observe that if the number 9 is placed in any small square other than x then we have a row or column that sums to at least 16.

A9 How many triangles (with positive area) are there which have their three corners as points chosen from the 2×3 grid shown?

A9
18



Solution: There must be exactly two vertices from either the top row or the bottom row. If there are two vertices from the top row, we could have AB, AC or BC, any of which could be matched with one of the three vertices in the bottom row. This gives $3 \times 3 = 9$ distinct triangles. By symmetry, using two vertices in the bottom row and one in the top row gives 9 triangles. Thus, there are $9 + 9 = 18$ triangles in total.

PART B: LONG ANSWER QUESTIONS

B1 An Egyptian grid is a square of numbers so that all numbers in the outside ring are 1's, all numbers in the next inner ring are 2's, all numbers in the next inner ring are 3's, and so forth. The following are the Egyptian grids of sizes $1, 2, 3, 4, 5, 6$, respectively. What is the sum of the entries of an Egyptian grid of size 9? The answer should be given as a whole number.

$$\begin{array}{c} 1 \end{array} \qquad \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \qquad \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{array} \qquad \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array} \qquad \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

Solution.
The answer is 165. One could brute force it or use $1^2 + 3^2 + 5^2 + 7^2 + 9^2$ or $4[8(1) + 6(2) + 4(3) + 2(4)] + 5 = 16(2 + 3 + 3 + 2) + 5 = 165$.

B2 Archibald runs round a 300 metre circular race track at 7 km/hr, while Beauregard runs at 8 km/hr. Suppose they start at the same time at the same place, but run in opposite directions.

(a) How long in minutes will it be before they first meet?

(b) If they keep running, will they ever meet at the point where they started, and if so, after how many minutes?

Solution.
Their relative speed is 15 kph, so they will cover 0.3 km in 0.3/15 hours, or 1.2 minutes, Archibald having covered 140 metres and Beauregard 160 metres. They will meet every 1.2 minutes, and $140n$ and $160n$ will be exact multiples of 300 just when $7n$ and $8n$ are multiples of 15. So they meet at their starting point for the first time, when $n = 15$, after $15 \times 1.2 = 18$ minutes.

B3 There are 2015 balls in 1000 boxes.

(a) Each box contains $1, 2$, or 3 balls.

(b) The number of boxes containing exactly one ball is greater than 308.

(c) The total number of balls in boxes containing more than one ball is greater than 1705.

How many boxes contain exactly 1, 2, and 3 balls, respectively?

Solution.

There are at least 309 boxes with exactly 1 ball, and at least 1706 balls in boxes containing 2 or 3 balls. This accounts for at least $309 + 1706 = 2015$ balls, which is all of them. So in fact there must be exactly 309 boxes containing one ball, and exactly 1706 balls in the other $1000 - 309 = 691$ boxes. If each of these 691 boxes contained just 2 balls, that would account for $2 \times 691 = 1382$ balls, leaving $1706 - 1382 = 324$ balls to be the third ball in some of these 691 boxes. So 309 boxes have just one ball, 324 boxes have three balls, and the remaining $691 - 324 = 367 (= 1000 - 309 - 324)$ boxes have two balls.

B4 A **preven number** is an integer that uses each digit in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ at most once, both starts and ends with a single digit that is prime or even, and each pair of consecutive digits forms a two-digit number which is prime or even.

For example, 8347 is preven since its first digit is even, its last digit is prime, and any two consecutive digits (83, 34, 47) are either even or prime. On the other hand, 8743 is not preven since 87 is neither even nor prime. The number 8343 is also not preven since it has a repeated digit.

(a) Find a four-digit preven number larger than 8347. The larger your four-digit preven number is, the more marks you may earn.

(b) Find a preven number which is as large as possible. The larger your number is, the more marks you may earn.

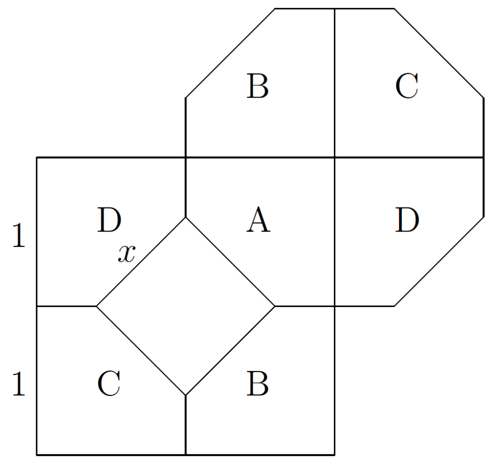
Solution.
For reference, the two-digit primes are

$$\{11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}.$$

(a) The largest is 8976, followed by 8974, 8973, 8972, 8967, 8964 and 8962.

(b) Any nine-digit preven number must begin with 5 since no two-digit prime or even number ends in 5. The largest possible is 598674312, followed by 598674132, 598674123, 598673412, 598673142 and 598673124.

B5 A square with edge length 2 is cut into five pieces: a square of edge length x, and four congruent pieces, A, B, C, and D which are reassembled to form an octagon which is regular, that is, has all its eight edges equal in length.



(a) What is x?

(b) Which piece has larger area: the square with edge length x or the piece labelled by A?

Solution.

(a) Four edges of the octagon (the sloping ones) are of length x. The other four are of length $2 - x\sqrt{2}$. Since the octagon is regular, we have $x = 2 - x\sqrt{2}$. Therefore,

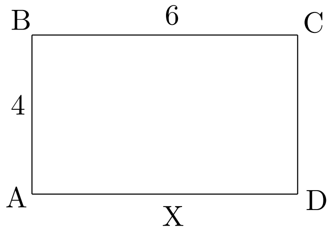
$$x = 2/(1+\sqrt{2}) = 2(\sqrt{2}-1) \approx 0.8284271.$$

(b) The area of one of the four congruent pieces is

$$\begin{aligned} \frac{1}{4}(4 - x^2) &= 1 - \left(\frac{x}{2}\right)^2 \\ &= 1 - (2 - 2\sqrt{2} + 1) \\ &= 1 - 2 + 2\sqrt{2} - 1 \\ &= 2(\sqrt{2} - 1) \\ &= x \end{aligned}$$

while the area of the square is x^2. Since $x < 1$, we have $x^2 < x$, so the square is smaller in area.

B6 Ellie is on her side of the tennis court (which is a 4 metres by 6 metres rectangle ABCD), practising serving from the midpoint X of the baseline AD. When there are three balls lying in her court she walks in straight lines to pick them up, from X to one ball, then to a second ball, then to the third ball and back to X. For example, if there were two balls at B and one at C, she could travel XBBCX for a total distance of 5+0+6+5=16 metres, or she could go XBCBX for a distance of 5+6+6+5=22 metres.



(a) Suppose the three balls are at points A, B and C. What is the shortest distance Ellie could walk to pick up the three balls? What is the longest distance Ellie could walk to pick up the three balls?

(b) Find places on the court for the three balls to be located so that the ratio

$$\frac{\text{longest distance Ellie could walk}}{\text{shortest distance Ellie could walk}}$$

is at least 1.5. The larger a ratio you find, the better your mark will be. (For extra credit, prove that your ratio is as large as possible.)

Solution.

(a) There are six paths to check: XABCX, XACBX, XBACX, XBCAX, XCABX, XCBAX. The shortest is length 18m. The longest is length $14 + 2\sqrt{13}$.

(b) We can get as close to 2 as we like by taking two balls close to A (or close to any other point) and one close to X.

To prove this, let XABCX be the minimum-length route, where X is Ellie's starting and ending point, and A,B,C are the positions of the three balls. Let XDEFX be the maximum-length route (or any route, actually), where A,B,C=D,E,F. Let e be one of the edges XD, DE, EF, FX. Then e is either (i) an edge of XABCX or (ii) connects X to B or A to C. If (i), then the length of e is at most half the length of XABCX. Suppose (ii), and that e connects X to B (wlog). Then the length of e is at most the smaller of the two paths XAB and XCB, and thus is at most half the length of XABCX. This holds for each of the four edges of XDEFX, thus the entire route XDEFX is at most twice as long as XABCX, as claimed.